Probability & Statistics (MA 2103)

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- **Instructors:** Dr. Teega Venkateswararao & Dr. Meraj Alam.
- Course Credits: **04** (3-1-0).
- Course Schedule (Per Week): **3** lectures + **1** tutorial.
- **Marks Breakup (tentative):**
  - **Minor I Exam: 20%.**
  - **Minor II Exam: 20%.**
  - **Major Exam: 40%.**
  - **Assignments & Quizzes: 15%.**
  - **Attendance: 5%.**

Syllabus

Topics include but are not restricted to:

- **Unit I. Probability and Random Variables:** Counting Techniques, Sample Space, Events, Axiomatic definition of probability, Conditional Probability, Independence of Events, Theorem of Total Probability, Bayes Theorem. Discrete and Continuous Random Variables, Probability Mass Function, Probability Density Function, Cumulative Distribution Function, Moments, Mathematical Expectation, Variance, Standard Deviation, Moment Generating Function, Characteristic Function.
- **Unit II. Discrete and Continuous Distributions:** Binomial Distribution, Geometric Distribution, Poisson Distribution, Uniform Distribution, Gamma Distribution, Exponential Distribution, Normal (Gaussian) Distribution, Markov's Inequality, Chebyshev's Inequality.
- **Unit III. Random Vectors:** Joint Probability Distribution of Functions of Random Variables, Independence of Random Variables, Covariance, Variance, Expectation, Correlation, Multinomial Distribution, Transformations of Random Variables, The Law of Large Numbers, The Central Limit Theorem.
- **Unit IV. Statistics:** Descriptive Statistics: Sample Mean, Sample Variance, Sample Standard Deviation and Sample Correlation Coefficient; Confidence Intervals, Parameter Estimation: Unbiasedness, Consistency, Point Estimator, Maximum Likelihood Estimators, Hypothesis Testing.

Text and Reference Books

- **Text books:**
  - **Sheldon M. Ross**, *Introduction to Probability and Statistics for Engineers and Scientists*, Elsevier, Sixth Edition, 2021.
  - **Sheldon M. Ross**, *A First Course in Probability*, Pearson, Ninth Edition, 2014.
  - **Alberto Leon-Garcia**, *Probability, Statistics, and Random Processes for Electrical Engineering*, Pearson, 2008.
- **Reference books:**
  - **Papoulis. A, and Unnikrishna Pillai. S**, *Probability, Random Variables and Stochastic Process*, Fourth Edition, McGraw-Hill, 2002.
  - **Hogg, Tanis and Rao**, *Probability and Statistical Inference*, Seventh Edition, Pearson, 2006.
  - **Sheldon M. Ross**, *Introduction to Probability Models*, Elsevier, Eleventh Edition, 2015.
  - **Jean Jacod and Philip Protter**, *Probability Essentials*, Springer, 2004.

Course Objectives

- **Probability Foundations:** Understand core concepts such as sample space, events, axiomatic probability, conditional probability, and independence.
- **Random Variables and Expectation:** Work with discrete and continuous random variables, and compute PMFs, PDFs, CDFs, expectations, and moments.
- **Probability Distributions:** Apply key discrete and continuous distributions, and use inequalities like Markov's and Chebyshev's.
- **Joint Distributions and Random Vectors:** Analyze joint distributions, compute covariance and correlation, and understand variable transformations.
- **Limit Theorems:** Comprehend the Law of Large Numbers and Central Limit Theorem and their role in statistical analysis.
- **Descriptive Statistics:** Compute and interpret sample mean, variance, standard deviation, and correlation.
- **Statistical Inference:** Construct confidence intervals, estimate parameters, and perform hypothesis testing using MLE and other methods.

Course Outcomes

Upon successful completion of the course, students will be able to:

- Apply probability concepts to model and solve real-world problems involving randomness and uncertainty.
- Define and analyze discrete and continuous random variables, and compute associated functions such as PMFs, PDFs, CDFs, and moments.
- Use standard probability distributions to model data and perform statistical reasoning.
- Analyze joint distributions of multiple random variables, and compute covariance and correlation.
- Apply limit theorems (LLN and CLT) for approximation.
- Compute and interpret descriptive statistics in the context of data analysis.
- Construct confidence intervals and perform parameter estimation using appropriate methods, including Maximum Likelihood Estimation.
- Perform hypothesis testing and draw valid conclusions from statistical data.

Course Policies

- **Attendance will be taken using the JUNO QR-Code in the first 10–15 minutes of the class. If you miss it, your attendance will not be updated later.**
- **You must attend enough classes to meet the minimum attendance rule. Please don't ask for exceptions at the end of the course.**
- **There will be no retests or second chances for quizzes or exams. Make sure you are present on those days.**
- **You will get enough time to finish and submit your assignments. No extra time will be given after the deadline.**
- **If you have questions or doubts, visit during office hours. There will be no doubt sessions in the week before the exam.**
- **The syllabus for each exam will be shared well in advance. So do not send emails asking about it.**
- **Every topic discussed in class and tutorials is important. Avoid asking whether a topic is "important for the exam" or not.**

Why Probability & Statistics?

- Every field, from delivery apps to rocket science, operates in an environment filled with **uncertainty**.
- Probability and Statistics help us measure, understand, and make decisions in the presence of that uncertainty.
- Consider these examples:
  - **Amazon** predicting delivery time during a festive rush faces uncertainty because sudden order surges, traffic delays, and warehouse bottlenecks can all affect the estimate.
  - **Uber** estimating arrival time must account for uncertainty caused by accidents, unpredictable traffic jams, and unexpected route changes.
  - **Zomato** ranking restaurants for fastest delivery deals with uncertainty from variations in kitchen preparation times, rider availability, and disruptions such as local events.
  - **Apple Watch** forecasting heart rate trends encounters uncertainty due to sudden bursts of physical activity, sensor inaccuracies, and daily lifestyle variations.
  - **Weather apps** predicting rainfall probabilities face uncertainty from rapid climate changes, incomplete atmospheric data, and measurement errors.

Your Journey Ahead

- This course will give you:
  - **Language to describe uncertainty**
  - **Tools to quantify and manage risk**
  - **Skills to model and analyze systems .**
  - **Statistical techniques to make informed decisions**
  - **Ability to connect theory to practice**
- By the end, you will be able to turn uncertain, incomplete data into clear, data-driven conclusions; a skill critical in engineering.

Let's start building your superpower.

The Sally Clark Case: A Lesson in Probability

Who was sally clark?

- Sally Clark was an English solicitor (Lawyer).
- In 1999, she was convicted of murdering her two infant sons.
- The case became famous for a major statistical error in the trial.
- It is a real world example of how misunderstanding probability can lead to serious injustice.

The Events

- Sally's first son died suddenly at 11 weeks old.
- The cause was recorded as Sudden Infant Death Syndrome (SIDS).
- About 14 months later, her second son also died under similar circumstances.
- The prosecution claimed it was extremely unlikely for two SIDS deaths to happen in the same family by chance.

The Statistical Claim

- An expert witness stated that the probability of one SIDS death in an affluent, non-smoking family was $\frac{1}{8543}$.
- He squared this probability to get:

$$\left(\frac{1}{8543}\right)^2 \approx \frac{1}{73\,000\,000}$$

- He presented this as the probability of two SIDS deaths in the same family.
- This number was portrayed as evidence that murder was overwhelmingly more likely.

The Errors in Reasoning

- **Independence Assumption:** The expert wrongly assumed the two deaths were independent events.
    - In reality, genetic or environmental factors could make a second SIDS death more likely if one has already occurred.
- **Prosecutor's Fallacy:** The expert confused the probability of two SIDS deaths ($P(\text{Evidence} \mid \text{Innocence})$) with the probability of innocence given the evidence ($P(\text{Innocence} \mid \text{Evidence})$).
- In other words, even if a rare event is unlikely, it doesn't automatically mean the alternative explanation (murder) is almost certain.

P(Evidence | Innocence) and P(Innocence | Evidence)

- Suppose **BOX-1** contains 100 red marbles.
- **BOX-2** contains 50 red marbles and 50 blue marbles.
- We randomly select one BOX and randomly select a marble from the selected BOX.
- P(Selecting red marble | BOX-1 is selected) $= 1 := 100\%$
- P(Selecting BOX-1 | Selected one is a red marble) $\neq 1 :\neq 100\%$
- P(Selecting BOX-1 | Selected one is a red marble) $:= 66\%$.

The Outcome

- Sally Clark was convicted in 1999 and served more than three years in prison.
- In 2003, her conviction was quashed after new medical evidence and recognition of the statistical errors.
- The case became a landmark example in law, statistics, and public policy.

Lessons for Probability and Statistics

- Misunderstanding probability can have life-altering consequences.
- Always question:
  - Are the events truly independent?
  - Am I interpreting conditional probabilities correctly?
- This course will give you the tools to avoid such mistakes and use statistics responsibly.

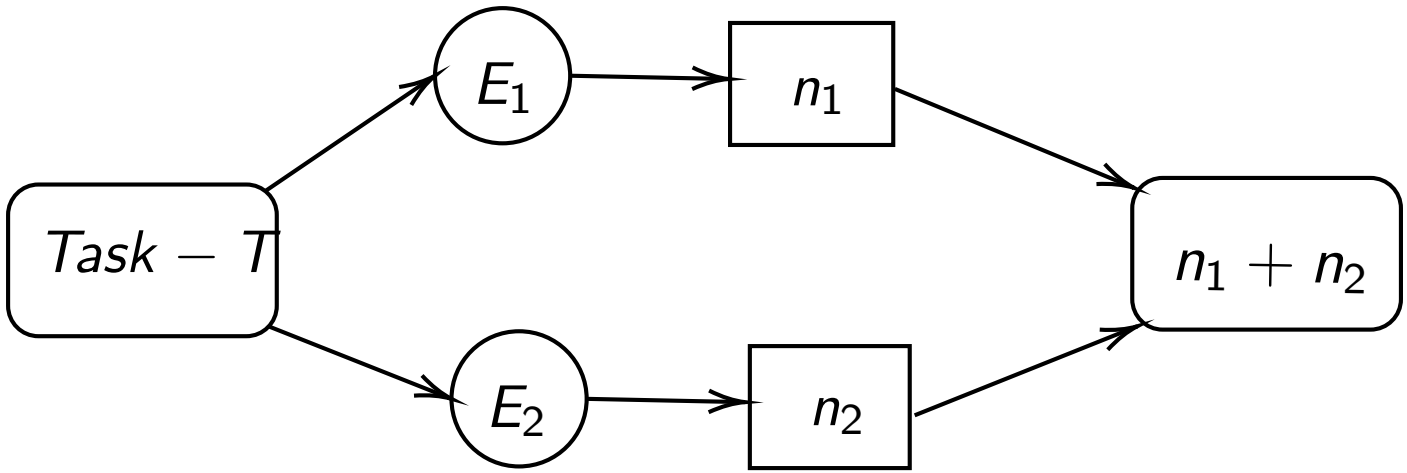
In statistics, numbers don't speak for themselves, we must interpret them correctly.

Counting

Basic counting principles

- **Sum rule**
- **Product rule**

- **Sum rule:** If a task can be done in either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.



Counting principles

Example 1: A streaming platform lists movies in three categories: Action (95 movies), Comedy (78 movies), and Drama (84 movies). No movie appears in more than one category. How many movies in total are there to choose from?

Answer: **95+78+84 = 257**

- **Product rule:** Suppose that a task (T) can be broken down into a sequence of two tasks (T_1 and T_2).
  - there are n_1 ways to do the first sub task T_1;
  - for each of these ways of doing the first task, there are n_2 ways to do the second sub task T_2;
  - then there are $n_1 n_2$ ways to do the task T.

Product rule

Example 2: Ravi wants to order a laptop bag from Amazon. Amazon gives him:

- Brand choice: American Tourister, Wildcraft, Skybags.
- Delivery speed: Standard (5 days), One-Day, Same-Day.

How many different ways can Ravi place his order?

T: Placing Order

T can be split into T_1 and T_2.

- T_1: Choose a brand $\rightarrow n_1 = 3$ ways
- T_2: Choose a delivery speed $\rightarrow n_2 = 3$ ways

Then:

$$\text{Total ways} = n_1 \times n_2 = 9.$$

Product Rule

Example 3: An Apple Watch generates an 8-character *Health Report ID* for each user. Each character can be:

- A lowercase English letter (a–z), OR
- A digit (0–9) — 10 choices.

Characters can repeat and can appear in any position. How many unique Health Report IDs are possible?

Solution:

- **Choices per character:** $26 + 10 = 36$ Each position can have 36 possible symbols.
- **Apply Product Rule:** The ID has 8 independent positions.

$$\text{Total IDs} = 36^8 = 2,821,109,907,456$$

Principle of inclusion -exclusion

- Let A and B be finite sets.
- There are $|A|$ and $|B|$ ways to select an element from A and B respectively.
- The number of ways to select an element from $A \cup B$ is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example 4: How many bit strings of length eight either start with a bit 0 or end with the two bits 11?

- A: Strings starting with 0:

$$0 \quad - \quad - \quad - \quad - \quad - \quad - \quad -$$

- $|A| = 2^7$.

- B: Strings ending with 11:

$$- \quad - \quad - \quad - \quad - \quad - \quad 1 \quad 1$$

- $|B| = 2^6$.
- $A \cap B$: Strings starting with 1 and ending with 00:

$$0 \quad - \quad - \quad - \quad - \quad - \quad 1 \quad 1$$

- $|A \cap B| = 2^5$.
- $A \cup B$: Strings starting with either 0 or ending with 11:

$$|A \cup B| = |A| + |B| - |A \cap B| = 128 + 64 - 32 = 160.$$

Example 5: How many positive integers not exceeding 1000 are divisible by 3 or divisible by 7.

Solution:

- A: Positive integers not exceeding 1000 and divisible by 3.

$$\left[\frac{1000}{3}\right] = 333$$

- $|A| = 333$.
- B: Positive integers not exceeding 1000 and divisible by 7.

$$\left[\frac{1000}{7}\right] = 142$$

- $|B| = 142$.
- $A \cap B$: Positive integers not exceeding 1000, divisible by 3 and 7.
- $|A \cap B| = \left[\frac{1000}{21}\right] = 47$.
- $A \cup B$: Positive integers not exceeding 1000 are divisible by 3 or divisible by 7.

$$|A \cup B| = |A| + |B| - |A \cap B| = 333 + 142 - 47 = 428.$$

THANK YOU.

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Principle of inclusion & exclusion (PIE)

- If A, B, and C are three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Problem 1: In a survey of 300 students, it was found that 150 students subscribed to **Netflix**, 120 to **Prime Video**, and 90 to **JioHotstar**. A total of 70 students subscribed to both **Netflix** and **Prime Video**, 50 subscribed to both **Netflix** and **JioHotstar**, and 40 subscribed to both **Prime Video** and **JioHotstar**. There were also 20 students who subscribed to all three services. How many students had at least one streaming subscription?

Solution: Let

$$|N| = 150, \ |P| = 120, \ \text{and} \ |J| = 90.$$

$$|N \cap P| = 70, \ |N \cap J| = 50, \ |P \cap J| = 40, \ \text{and} \ |N \cap P \cap J| = 20.$$

Use the PIE formula:

$$|N \cup P \cup J| = |N| + |P| + |J| - |N \cap P| - |N \cap J| - |P \cap J| + |N \cap P \cap J|.$$

$$|N \cup P \cup J| = 150 + 120 + 90 - 70 - 50 - 40 + 20 = 220.$$

Therefore, $\boxed{220}$ students have at least one subscription.

Principle of Inclusion & Exclusion

For 2 sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For 3 sets A, B, and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

For n sets $A_1, A_2, \ldots, A_n$:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Idea: Add sizes of all sets, subtract overlaps, add triple overlaps, subtract quadruple overlaps, and so on.

Permutations & Combinations

Permutation: A permutation is an arrangement of objects in a specific order.

Examples:

- An engineer is creating an 8-character password for securing a cloud server. The password can only use uppercase letters (A–Z) and digits (0–9).
- A drone delivery system needs to deliver packages to 5 different locations. The number of different delivery sequences are permutations.
- An airline is assigning 5 pilots to 5 different flights for the day. The assignment order is critical for scheduling.

Permutations

- Consider a set A containing n objects.
- How many number of permutations of size (order) r can be formed using the objects of A?
  - For the first position there are n choices.
  - For each choice in the first position, there are $n-1$ choices for the second position.
  - Similarly, for the rth position there are $(n-(r-1))$.

$$n.(n-1).(n-2)...(n-(r-1))$$

- $n.(n-1).(n-2)...(n-(r-1)) = \frac{n!}{(n-r)!} = {}^nP_r$
- ${}^nP_n = n!$, and ${}^nP_0 = 1$

Permutations

Problem 2: A robotics competition has 8 finalist teams: Alpha, Beta, Cyborg, Delta, Echo, Falcon, Gamma, and Hyper. If Alpha and Beta must be placed together as a pair in the ranking list, how many different rankings are possible?

Solution:

We can treat **Alpha and Beta** as a single team ("super-team") since they must be together.

- This reduces the number of "objects" to arrange from 8 to 7 (the pair + the other 6 teams).
- The 7 objects can be arranged in:

$$7! \text{ ways.}$$

- Within the "super-team", Alpha and Beta can be arranged in:

$$2! \text{ ways.}$$

- Therefore, the total number of different rankings is:

$$\text{Total ways} = 7! \times 2! = 5040 \times 2 = 10080.$$

Permutations

Problem 3: In how many ways can the batting lineup be arranged for a cricket team of 11 players?

Solution: 11!

Problem 4: The Probability & Statistics course consists of 500 boys and 450 girls. If the students are ranked according to their performance. Assume that no two students obtain the same score.

- How many different rankings are possible?
- If the boys are ranked just among themselves and the girls just among themselves, how many different rankings are possible?

Solution: **(a)** There are 950 students in total. The number of possible rankings is:

$$950!.$$

(b) The boys can be ranked among themselves in 500! ways, and the girls in 450! ways. Thus,

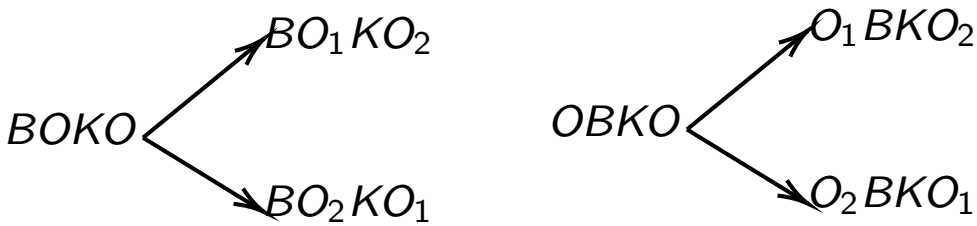
$$500!450!.$$

Permutations

Problem 5: How many different letter arrangements can be formed from the letters **BOOK**?

Solution: The letters are: B, O_1, O_2, K. To begin with, we treat O_1 and O_2 as distinct. In this scenario, the possible orderings are

BO_1O_2K	BO_1KO_2	BO_2O_1K	BO_2KO_1	BKO_1O_2	BKO_2O_1
O_1BO_2K	O_1BKO_2	O_1O_2BK	O_1O_2KB	O_1KBO_2	O_1KO_2B
O_2BO_1K	O_2BKO_1	O_2O_1BK	O_2O_1KB	O_2KBO_1	O_2KO_1B
KBO_1O_2	KBO_2O_1	KO_1BO_2	KO_1O_2B	KO_2BO_1	KO_2O_1B



If O_1 and O_2 are considered identical, each arrangement above is counted twice. Therefore, the total number of distinct arrangements is:

$$\frac{4!}{2!} = \frac{24}{2} = 12.$$

Permutations

Problem 6: How many different letter arrangements can be formed from the letters of the word **BALLOON**?

Solution: We first note that there are 7! permutations of the letters $B\,A\,L_1\,L_2\,O_1\,O_2\,N$ when the two L's and two O's are distinguished from each other.

However, consider any one of these permutations, for instance, $L_1\,L_2\,O_1\,B\,O_2\,N\,A$.

If we now permute the L's among themselves and the O's among themselves, the resultant arrangement would still be of the form $L\,L\,O\,B\,O\,N\,A$.

That is, for each distinct arrangement, the following permutations correspond to the same arrangement:

$$\begin{aligned} L_1 L_2 \quad &\leftrightarrow \quad L_2 L_1, \\ O_1 O_2 \quad &\leftrightarrow \quad O_2 O_1. \end{aligned}$$

So, the total number of arrangements is obtained by dividing 7! by the product $2! \cdot 2!$:

$$\frac{7!}{2! \cdot 2!} = \frac{5040}{4} = 1260.$$

Permutations & Combinations

- The number of different permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike is

$$\frac{n!}{n_1!n_2!...n_r!}.$$

Problem 7: In a chess tournament there are 12 competitors representing different countries. 3 players from India, 2 from Russia, 2 from China, 2 from the United States, 2 from Norway, and 1 from Spain. If the official result records only the nationalities of the players in the order in which they finished, how many distinct outcome lists of nationalities are possible?

Solution:

$$\frac{12!}{3!2!2!2!2!1!}.$$

Problem 8: Suppose you have intercepted an encrypted file protected by a password with these clues: the password definitely contains 3 '9's, 2 'z's, 3 '@' symbols, and 2 '$' signs, arranged in some order. How many different password combinations you might have to test to guarantee access?

Solution:

$$\frac{10!}{3!2!3!2!} = 25,200.$$

Combinations

Combinations: A combination is a selection of objects where the order does not matter.

Examples:

- Choosing 11 players from 15 candidates for a Test match.
- Selecting a meal: lets you choose 3 items from 10 options (sandwich, drink, snack).
  - A "sandwich+drink+chips combo" is the same as chips+drink+sandwich.
- Testing 30 random smartphones from a production batch of 1000.
  - The testing sequence doesn't affect defect detection.
- A data scientist has 12 potential predictor variables (e.g., age, income, location, education, credit score, etc.). Wants to test all possible subsets of 3 features to determine which combination best predicts the target (e.g., loan approval).
  - Machine learning models treat {age, income, location} identically to {location, age, income}

Combinations

- Consider a set A containing n objects.
- The number of combinations of size (order) r can be formed using the objects of A is

$$\binom{n}{r} = \frac{n!}{(n-r)!}.\frac{1}{r!} = \#\,(\text{Permutations of size r}).\frac{1}{r!} = \frac{{}^{n}P_{r}}{r!}.$$

- It is equivalent to selecting a subset of size r from a set containing n elements.

Problem 9: A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Solution:

$$\binom{20}{3} = \frac{20!}{17!3!} = 1140.$$

Problem 10: Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution:

$$- \; W \; - \; W \; - \; W \; - \; W \; - \; W \; - \; W \; - \; W \; - \; W \; -$$

- If no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna.
- $n-m+1$ positions are available to place defective antennas.
- Hence, the number of linear orderings are there in which no two defectives are consecutive is

$$\binom{n-m+1}{m}.$$

Permutations & Combinations

Problem 11: During an award ceremony, 20 athletes are to be seated in a single row. Among them are Rahul and Sameer. In how many possible seating arrangements are there exactly 5 other athletes seated between Rahul and Sameer?

Solution:

- First select 5 people from 18, and that can be done in $\binom{18}{5}$.
- These 5 people can arranged among themself in 5! ways.
- Rahul on the left and Sameer on the right, or Sameer on the left and Rahul on the right. Hence, 2! arrangements.
- We treat the pair (Rahul, Sameer) with exactly 5 people in between as a single *block* of 7 seats (Rahul + 5 people + Sameer).
- The remaining 13 athletes (outside the block) + block can be arranged in 14! ways.

Therefore, the total number of arrangements is:

$$\binom{18}{5} \times 2 \times 5! \times 14!.$$

Identities

- $$\binom{n}{r} = \binom{n}{n-r}, \quad r \leq n.$$
- Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}, \quad x, y \in \mathbb{R}, \quad n \in \mathbb{N}.$$

- $$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$
- $$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0.$$
- $$\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n.$$

Identities

- Pascal's identity:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}, \quad r \leq n.$$

- Vandermonde's Identity:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}, \quad r \leq n, \quad r \leq m.$$

- $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

THANK YOU.

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Permutations with repetitions

- How many number of ways are there to distribute r different objects into n different boxes, if no restriction is placed on the number of objects permitted in a box?

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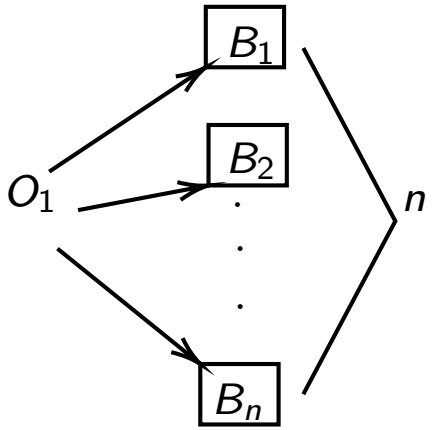

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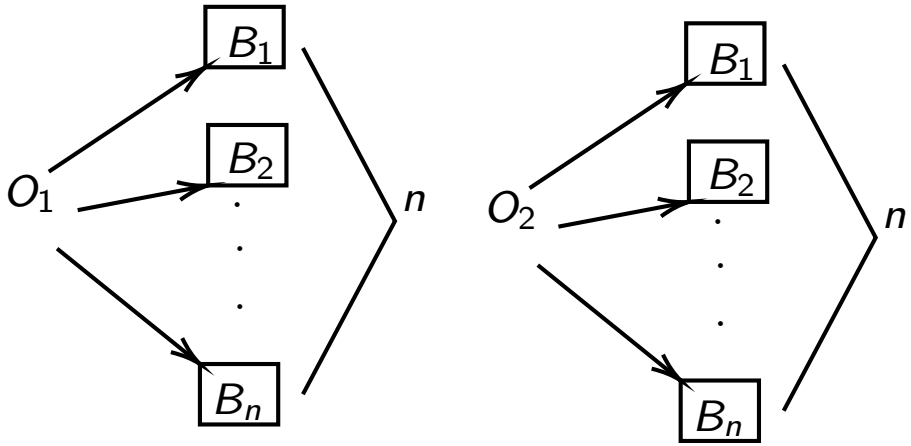

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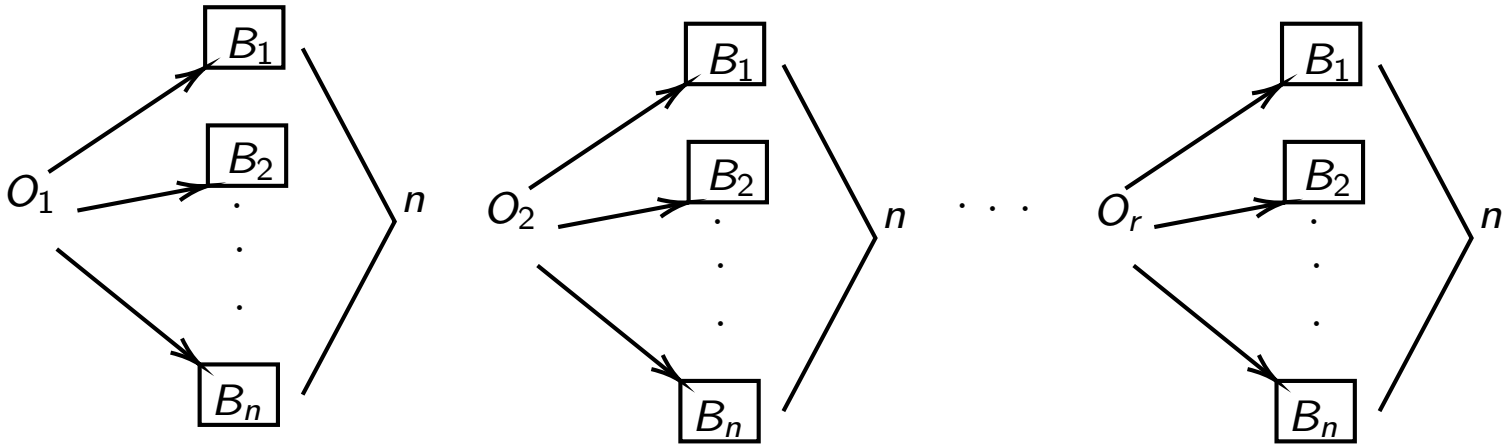

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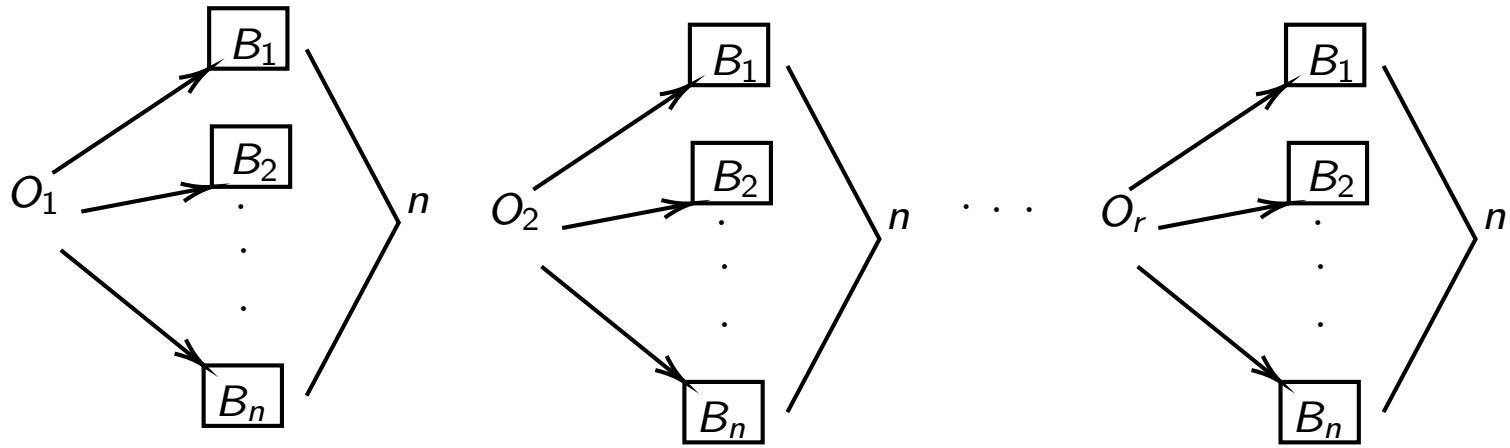

- **Answer:** n^r.

Permutations with repetitions

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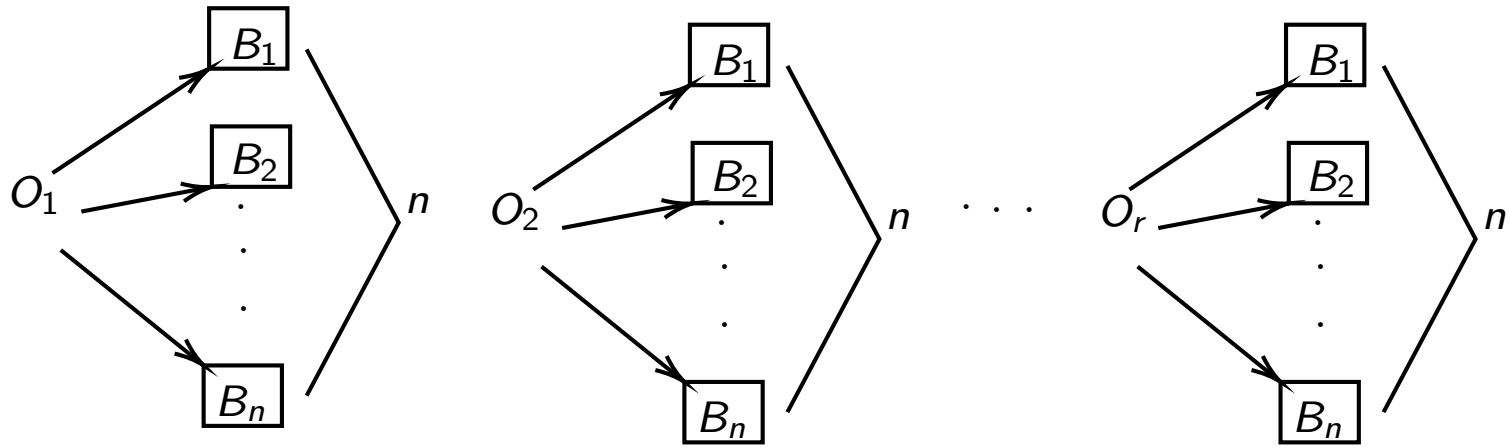

- **Answer:** n^r.

Table: Permutations and combinations formulas.

n, r	W.O.R	W.R
Permutation	${}^nP_r = \frac{n!}{(n-k)!}$	n^r
Order does not matter	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	??

Combinations with repetitions

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  - {25 Black, 20 White, 5 Red}

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Combinations with repetitions

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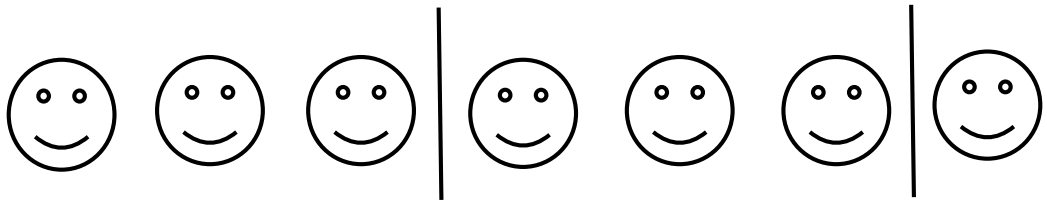

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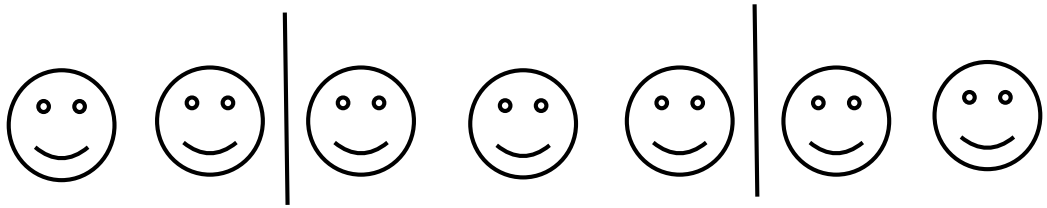

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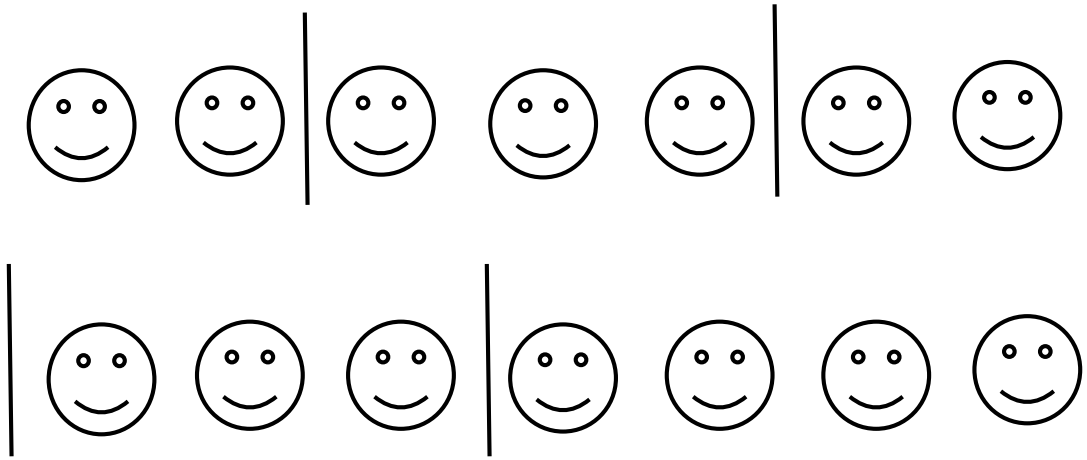

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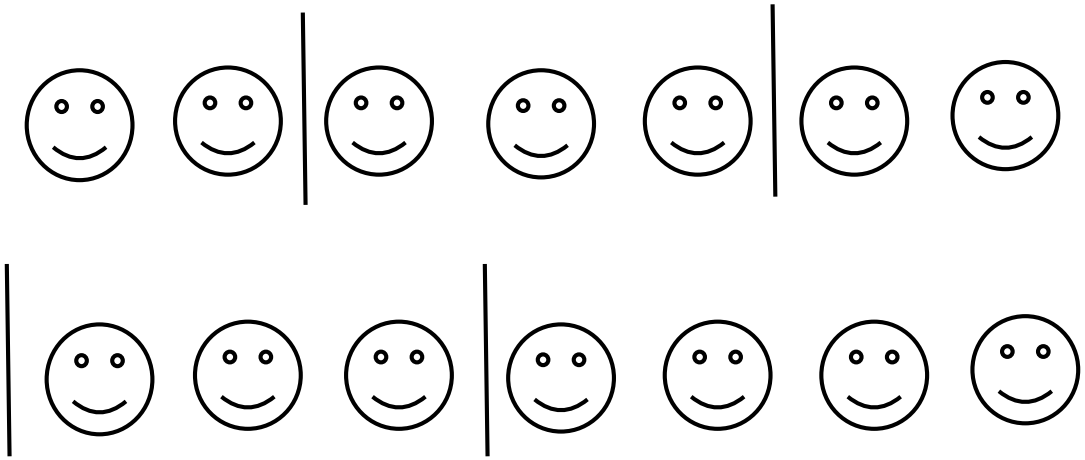

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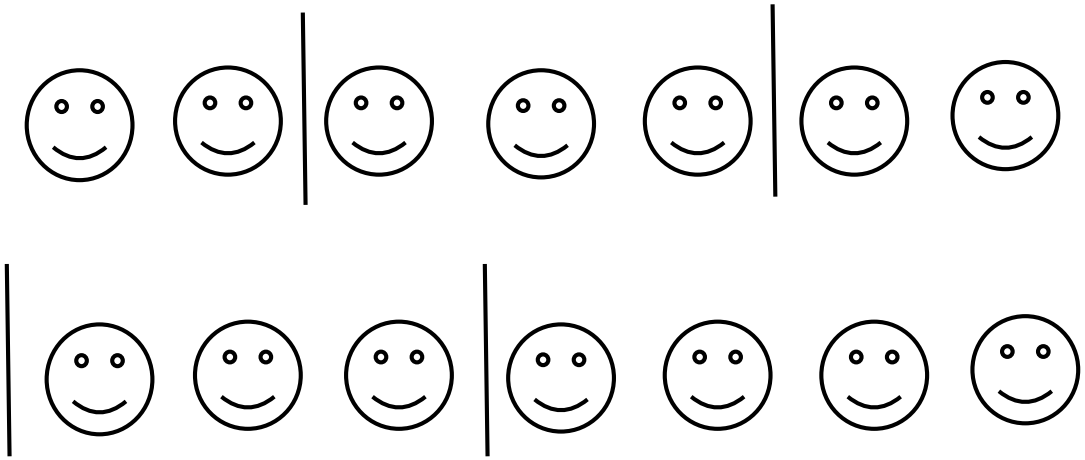

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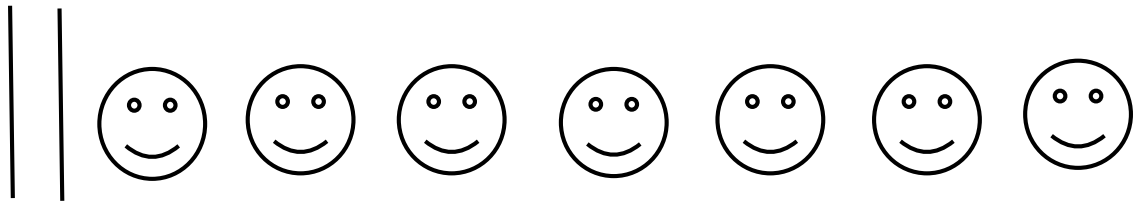

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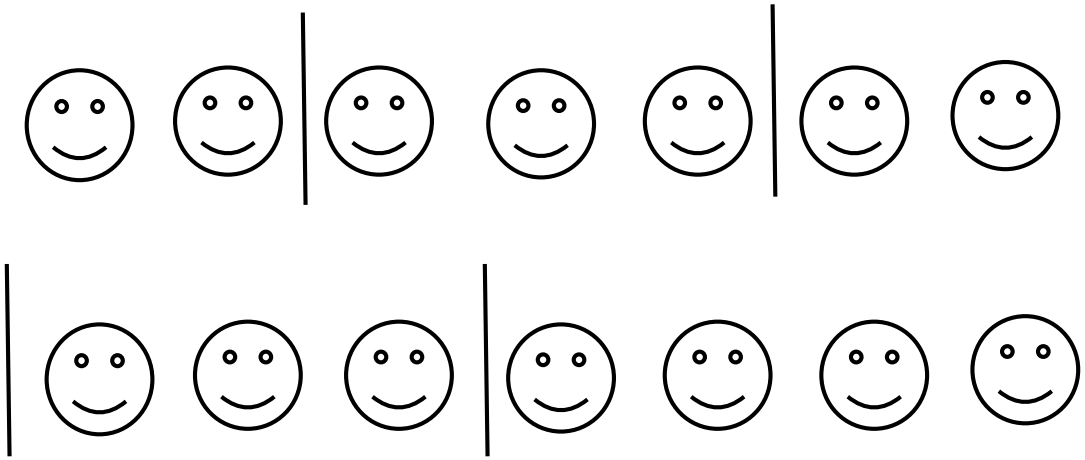

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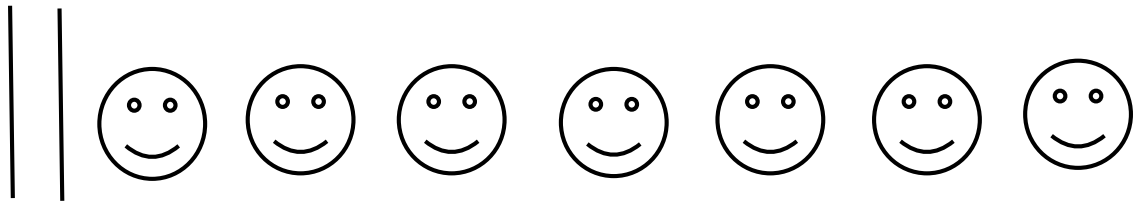

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Combinations with repetitions

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We perform a change of variables,

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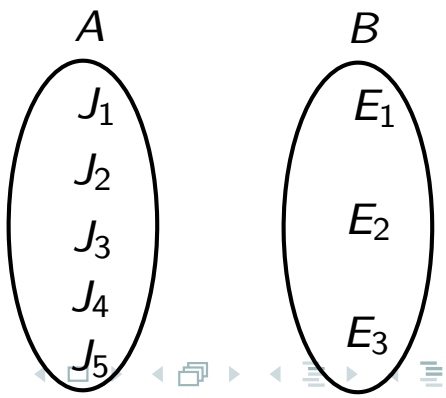

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- Hence, the total number of assignments is $3^5 - |F_1 \cup F_2 \cup F_3|$.
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- Therefore, the answer is

$$243 - \{2^5 + 2^5 + 2^5 - 1 - 1 - 1 + 0\} = 150.$$

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Solution: Let

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- Let F_i : set if functions from A to B, and E_i is not contained in the range of these functions, $i = 1, 2, ..., n$.
- Number of on-to functions: $n^m - |F_1 \cup F_2 \cup ... \cup F_n|$.

Number of on-to Functions

Problem: 6 How many on-to functions are there from the the set containing m elements to a set containing n ($m \geq n$).

Solution: Let

$$A = \{J_1, J_2, ..., j_m\} \text{ and } B = \{E_1, E_2, ..., E_n\}$$

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- Number of on-to functions: $n^m - |F_1 \cup F_2 \cup ... \cup F_n|$.
- $|F_i| = (n-1)^m$, $i = 1, 2, ..., n$. $|F_i \cap F_j| = (n-2)^m$, $i, j = 1, 2, ..., n$, $i \neq j$.

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- $|F_i| = (n-1)^m$, $i = 1, 2, ..., n$. $|F_i \cap F_j| = (n-2)^m$, $i, j = 1, 2, ..., n$, $i \neq j$.
- $|F_i \cap F_j \cap F_k| = (n-3)^m$, $i, j, k = 1, 2, ..., n$, $i \neq j \neq k$.

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- $|F_i \cap F_j \cap F_k| = (n-3)^m$, $i, j, k = 1, 2, ..., n$, $i \neq j \neq k$.
- $|F_1 \cap F_2 \cap ... \cap F_n| = 0$.

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- $|F_i \cap F_j \cap F_k| = (n-3)^m$, $i, j, k = 1, 2, ..., n$, $i \neq j \neq k$.
- $|F_1 \cap F_2 \cap ... \cap F_n| = 0$.
- $|F_1 \cup F_2 \cup ... \cup F_n| = \binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - ... + (-1)^n \binom{n}{n-1} 1^m$.

Number of on-to Functions

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Solution: Let

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- $|F_i| = (n-1)^m$, $i = 1, 2, ..., n$. $\quad |F_i \cap F_j| = (n-2)^m$, $i, j = 1, 2, ..., n$, $i \neq j$.
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- $|F_1 \cap F_2 \cap ... \cap F_n| = 0$.
- $|F_1 \cup F_2 \cup ... \cup F_n| = \binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - ... + (-1)^n \binom{n}{n-1} 1^m$.
- Hence, number of on-to functions is

$$n^m - \left\{ \binom{n}{1}(n-1)^m - \binom{n}{2}(n-2)^m + \binom{n}{3}(n-3)^m - ... + (-1)^n \binom{n}{n-1} 1^m \right\}.$$

Hat-Problem

Problem 7: Suppose there are 5 people at a party, and each of them checks in a hat at the entrance. When they leave, the hats are returned at random, with each person receiving exactly one hat. In how many different ways can the hats be returned so that no person gets his/her own hat back?

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- Number ways: $5! - |P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5|$.

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- $|P_i \cap P_j \cap P_k \cap P_l| = (5-4)!$, $i, j, k, l = 1, 2, 3, 4, 5$, $i \neq j \neq k \neq l$.

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- $|P_i| = (5-1)!$, $i = 1, 2, 3, 4, 5$. $\quad |P_i \cap P_j| = (5-2)!$, $i, j = 1, 2, 3, 4, 5$, $i \neq j$.
- $|P_i \cap P_j \cap P_k| = (5-3)!$, $i, j, k = 1, 2, 3, 4, 5$, $i \neq j \neq k$.
- $|P_i \cap P_j \cap P_k \cap P_l| = (5-4)!$, $i, j, k, l = 1, 2, 3, 4, 5$, $i \neq j \neq k \neq l$.
- $|P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5| = (5-5)!$.

Hat-Problem

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- $|P_i \cap P_j \cap P_k \cap P_l| = (5-4)!$, $i, j, k, l = 1, 2, 3, 4, 5$, $i \neq j \neq k \neq l$.
- $|P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5| = (5-5)!$.
- $|P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5| = \binom{5}{1}(5-1)! - \binom{5}{2}(5-2)! + \binom{5}{3}(5-3)! - \binom{5}{4}(5-4)! + \binom{5}{5}(5-5)!$.

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- $|P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5| = (5-5)!$.
- $|P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5| = \binom{5}{1}(5-1)! - \binom{5}{2}(5-2)! + \binom{5}{3}(5-3)! - \binom{5}{4}(5-4)! + \binom{5}{5}(5-5)!$.
- Hence, number ways is

$$5! - \left\{ \binom{5}{1}(5-1)! - \binom{5}{2}(5-2)! + \binom{5}{3}(5-3)! - \binom{5}{4}(5-4)! + \binom{5}{5}(5-5)! \right\}.$$

Hat-Problem

Problem 8: Suppose there are n people at a party, and each of them checks in a hat at the entrance. When they leave, the hats are returned at random, with each person receiving exactly one hat. In how many different ways can the hats be returned so that no person gets his/her own hat back?

Hat-Problem

Problem 8: Suppose there are n people at a party, and each of them checks in a hat at the entrance. When they leave, the hats are returned at random, with each person receiving exactly one hat. In how many different ways can the hats be returned so that no person gets his/her own hat back?

solution:

$$n! - \left\{ \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^{n+1}\binom{n}{n}(n-n)! \right\}$$

Hat-Problem

Problem 8: Suppose there are n people at a party, and each of them checks in a hat at the entrance. When they leave, the hats are returned at random, with each person receiving exactly one hat. In how many different ways can the hats be returned so that no person gets his/her own hat back?

solution:

$$n! - \left\{ \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \ldots + (-1)^{n+1}\binom{n}{n}(n-n)! \right\}$$

$$= n! - n!\left\{1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n+1}\frac{1}{n!}\right\}.$$

THANK YOU.

Probability & Statistics (MA 2103)

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Random Experiment

A **random experiment** is an experiment that satisfies the following conditions:

- The set of all possible outcomes is known in advance.
- The exact outcome of any single trial cannot be predicted beforehand.
- The experiment can be repeated under identical conditions.

Sample Space: The collection of all possible outcomes of a random experiment is called the sample space. A sample space will usually be denoted by Ω.

- R_1**:** Count the number of cars passing through a traffic light in 1 minute.
  - $\Omega_1 = \{0, 1, 2, 3, \dots\}$.
- R_2**:** A visitor enters an e-commerce site; do they buy or not?
  - $\Omega_2 = \{\text{Buy}, \text{ Not Buy}\}$.
- R_3**:** Number of pizzas ordered in a restaurant during dinner time.
  - $\Omega_3 = \{0, 1, 2, 3, \dots\}$.

Random experiments

Event: Any subset of a sample space is an event.

- Consider a random experiment of rolling a six faced die. The sample space is $\{1, 2, 3, 4, 5, 6\}$.
  - $E_1 = \{2, 4, 6\}$ (Outcome is even).
  - $E_2 = \{3, 6\}$ (Outcome is a multiple of 3).
- **If the outcome of the random experiment is a member of the set $E \subseteq \Omega$, we say that the event E has occurred.**
- Two events $A \subseteq \Omega$ and $B \subseteq \Omega$ are said to be mutually exclusive if they cannot occur simultaneously, i.e., if $A \cap B = \phi$.
  - In the random experiment of rolling a die, the events $A = \{1, 3, 5\}$ (Out come is even) and $B = \{2, 4, 6\}$ (Outcome is even) are mutually exclusive.
  - Consider random experiment of drawing a card from a standard deck of 52 playing cards. $A = \{$King of Hearts, King of Diamonds, King of Clubs, King of Spades$\}$ (Drawing a King card from the deck), $B = \{$Queen of Hearts, Queen of Diamonds, Queen of Clubs, Queen of Spades$\}$ (Drawing a Queen card from the deck). Here, the events A and B are mutually exclusive.

Classical Method

- This method for assigning probabilities is suited to experiments with a finite collection of outcomes, provided that every outcome is equally likely.
- Consider a finite sample space, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, $|\Omega|$ is finite.
- Let E be a subset of Ω. The number of elements in E is denoted by $|E|$.
- An outcome $\omega \in \Omega$ is said to be *favorable* to the event E if $\omega \in E$.
- The probability of an event E is then defined as

$$\mathbb{P}(E) = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}} = \frac{|E|}{|\Omega|} = \frac{|E|}{n}.$$

- Note that probabilities assigned through the classical method satisfy the following properties:
  - For any event E, $\mathbb{P}(E) \geq 0$.
  - For mutually exclusive events E_1 and E_2, $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$.
  - $\mathbb{P}(\Omega) = 1$.

Relative frequency method

- Suppose that we have independent repetitions of a random experiment under identical conditions (independent repetitions means that the outcome of one trial is not affected by the outcome of another trial).
- Let $f_N(E)$ denote the number of times an event E occurs (also called the frequency of event E in N trials) in the first N trials.
- Let $r_N(E) = \dfrac{f_N(E)}{N}$ be denote the corresponding relative frequency.
- The relative frequency method of assigning probabilities the probability of an event E is given by

$$\mathbb{P}(E) = \lim_{N\to\infty} r_N(E) = \lim_{N\to\infty} \frac{f_N(E)}{N}.$$

- Note that probabilities assigned through the Relative frequency method also satisfy the following properties:
  - For any event E, $\mathbb{P}(E) \geq 0$.
  - For mutually exclusive events E_1 and E_2, $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$.
  - $\mathbb{P}(\Omega) = 1$.

Problem 1

If 3 numbers are randomly selected from the set $\{1, 2, \ldots, 10\}$, what is the probability that the selected numbers are in arithmetic progression (AP)?

Solution:

- Total number of ways to select 3 numbers: $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120 = |\Omega|$.
- E: Numbers are in AP.
- Favorable outcomes (numbers in AP):

  Let a be the middle term and $d > 0$ the common difference. The numbers in AP are $a - d, a, a + d$. To stay within $\{1, 2, \ldots, 10\}$, we require:

  $$1 \leq a - d \quad \text{and} \quad a + d \leq 10 \implies d \leq \min(a - 1, 10 - a)$$

  The number of sequences for each middle term a is:

a	Number of sequences	a	Number of sequences
1	0	6	4
2	1	7	3
3	2	8	2
4	3	9	1
5	4	10	0

- $\mathbb{P}(E) = \frac{\text{favorable outcomes}}{\text{total outcomes}} = \frac{20}{120} = \frac{1}{6}$.

Simulation of Arithmetic Progressions

```
import random
import matplotlib.pyplot as plt

# Function to check if three numbers form an arithmetic progression
def is_ap(triple):
    triple_sorted = sorted(triple)
    return (triple_sorted[1] - triple_sorted[0]) == (triple_sorted[2] - triple_sorted[1])

# Simulation parameters
max_trials = 20000
results = []

success_count = 0
for n in range(1, max_trials + 1):
    sample = random.sample(range(1, 11), 3)    # pick 3 numbers without replacement
    if is_ap(sample):
        success_count += 1
    results.append(success_count / n)          # running proportion

# Plotting
plt.figure(figsize=(8,5))
plt.plot(range(1, max_trials+1), results, label="Simulated Proportion")
plt.axhline(1/6, color='red', linestyle='--', label="Theoretical Probability (1/6)")
plt.xlabel("Number of Trials")
plt.ylabel("Proportion of APs")
plt.title("Probability of 3 numbers (1–10) forming an Arithmetic Progression")
plt.legend()
plt.show()
```

Birth day Paradox

Problem 2: If n (< 365) people are present in a room, what is the probability that **at least two of them celebrate their birthday on the same day of the year**? Assume a year has 365 days and all birthdays are equally likely.

Solution:

- Let E be the event that at least two people share a birthday. It is easier to first consider the complement event

$$E^c = \text{No two people share a birthday.}$$

  Then, $\mathbb{P}(E) = 1 - \mathbb{P}(E^c)$.
- If no two people share a birthday, the number of favorable outcomes is

$$365 \cdot 364 \cdot 363 \cdots (365 - n + 1) = \frac{365!}{(365 - n)!}.$$

  and the total number of possible birthday assignments is 365^n. Thus,

$$\mathbb{P}(E^c) = \frac{365!}{(365 - n)!\, 365^n}.$$

- Probability of at least two people sharing a birthday:

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \frac{{}^{365}P_n}{365^n}.$$

Birth day Paradox

- For $n = 23$:

$$\mathbb{P}(E) \approx 1 - \exp\Big(-\frac{23 \cdot 22}{2 \cdot 365}\Big) = 1 - \exp(-0.693) \approx 0.50.$$

- For $n = 50$:

$$\mathbb{P}(E) \approx 1 - \exp\Big(-\frac{50 \cdot 49}{2 \cdot 365}\Big) = 1 - \exp(-3.356) \approx 0.965.$$

- For $n = 100$:

$$\mathbb{P}(E) \approx 1 - \exp\Big(-\frac{100 \cdot 99}{2 \cdot 365}\Big) = 1 - \exp(-13.56) \approx 0.9999987.$$

Simulation of Birthday Paradox

```
import random
import matplotlib.pyplot as plt

# Function to check if at least 2 people share a birthday
def has_duplicate(birthdays):
    return len(birthdays) != len(set(birthdays))

# Simulation parameters
group_size = 23
trials = 10
results = []

success_count = 0
for n in range(1, trials + 1):
    birthdays = [random.randint(1, 365) for _ in range(group_size)]
    if has_duplicate(birthdays):
        success_count += 1
    results.append(success_count / n)   # running proportion

# Plotting
plt.figure(figsize=(8,5))
plt.plot(range(1, trials+1), results, label="Simulated Proportion")
plt.axhline(0.5, color='red', linestyle='--', label="Reference Line (0.5)")
plt.xlabel("Number of Trials")
plt.ylabel("Proportion with Shared Birthday")
plt.title(f"Birthday Paradox Simulation (Group of {group_size})")
plt.legend()
plt.show()
```

THANK YOU.

Probability & Statistics (MA 2103)

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Problem 1

In a group of 30 friends on a messaging app, one person (the *originator*) creates a post and forwards it to another friend. That friend then forwards it to a third friend, and so on. At each step, the sender randomly selects one of the other 29 friends (not including them self) as the next recipient. Suppose the post is forwarded **15 times**. Find:

- The probability that the post is never forwarded back to the **originator** during these 15 forwards.
- The probability that no friend receives the post **more than once** (i.e., all 15 recipients are distinct individuals).

Solution:

- Probability that the originator never receives the post:

  At each step, the sender must avoid sending the post back to the originator. There are 28 choices out of 29 friends:

$$\mathbb{P}(\text{Originator never receives post}) = \frac{29.28^{14}}{29^{15}} \approx 0.592.$$

- Let $R_1, R_2, \ldots, R_{15}$ be the recipients.

$$\mathbb{P}(\text{All 15 recipients are distinct}) = \frac{29 \cdot 28 \cdot 27 \cdots 15}{29^{15}} \approx 0.207.$$

Problem 2

Problem. There are five empty boxes. Balls are placed independently one after another in randomly selected boxes. What is the probability that the *fourth* ball is the first to be placed in an occupied box?

Solution: Let a length-4 sequence denote the boxes chosen by the four balls.

- Total sequences: $|\Omega| = 5^4$.
- E: The *fourth* ball is the first to be placed in an occupied box.
- Favorable sequences: the first three choices are all distinct, and the fourth choice hits one of those occupied boxes.

$$\#(\text{first three distinct}) = 5 \times 4 \times 3,$$
$$\#(\text{fourth hits an occupied box}) = 3.$$

Hence

$$\#(\text{favorable}) = (5 \cdot 4 \cdot 3) \cdot 3 = 180.$$

Therefore,

$$\mathbb{P}(E) = \frac{\#(\text{favorable})}{|\Omega|} = \frac{180}{5^4}.$$

Problem 3

Problem. Two hundred tickets are numbered $1, 2, \ldots, 200$ and arranged at random. Five tickets are given, one each, to persons A, B, C, D, E. What is the probability that A gets the *largest* (among these five) and E gets the *smallest*?

Solution:

- Total assignments: $|\Omega| = \binom{200}{5} \cdot 5!$.
- M: A gets largest and E smallest numbered ticket.
- *Favorable assignments:* force the largest to A and the smallest to E; the remaining three tickets can be assigned freely to B, C, D in 3! ways. Thus,

$$\#(\text{favorable}) = \binom{200}{5} \cdot 3!.$$

Therefore,

$$\mathbb{P}(A \text{ gets largest and } E \text{ smallest numbered ticket}) = \mathbb{P}(M) = \frac{\binom{200}{5} \cdot 3!}{\binom{200}{5} \cdot 5!} = \frac{1}{20}.$$

Problem 4

From the six letters $\{A, B, C, D, E, F\}$, three letters are chosen at random *with replacement*. What is the probability that either the word **BAD** or the word **CAD** can be formed from the chosen letters? .

Solution:

- Total number of ordered outcomes (three draws with replacement): $|\Omega| = 6^3 = 216$.
- E: Either the word **BAD** or the word **CAD**.
- To be able to form the word `BAD` the three drawn letters must be exactly the $\{B, A, D\}$. Since these letters are distinct, any permutation of them in the ordered draws works. Number of ordered sequences that give these three letters: $3! = 6$. Hence

$$\mathbb{P}(\text{can form BAD}) = \frac{6}{216} = \frac{1}{36}.$$

- Similarly for `CAD`:

$$\mathbb{P}(\text{can form CAD}) = \frac{6}{216} = \frac{1}{36}.$$

- The events "can form BAD" and "can form CAD" are disjoint, so add probabilities:

$$\mathbb{P}(E) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}.$$

Electricity Consumption in a City

- Suppose we have data of electricity consumption (in units per month) for all households in the city.
- $$\Omega = \{\text{All households in the city}\}.$$
- For each household $\omega \in \Omega$, we record the number of electricity units consumed in a given month.
- $$A = \{\omega \in \Omega : \text{household } \omega \text{ consumes more than 2000 units in a month}\}.$$
- $$B = \{\omega \in \Omega : \text{household } \omega \text{ consumes less than 100 units in a month}\}.$$
- $$A \cap B = \{\omega \in \Omega : \omega \text{ consumes more than 200 units and less than 100 units}\} = \varnothing,$$
- $$A^c = \{\omega \in \Omega : \text{household consumes at most 2000 units}\},$$
- $$B^c = \{\omega \in \Omega : \text{household consumes at least 100 units}\}.$$

Definition of a σ-field

Let Ω be a non-empty set (called the **sample space**). A collection $\mathcal{F}$ of subsets of Ω is called a σ-**field** (or σ-**algebra**) if it satisfies the following properties: 0.2cm

- $\Omega \in \mathcal{F}$.
- If $A \in \mathcal{F}$, then $A^c = \Omega \setminus A \in \mathcal{F}$
- If $A_1, A_2, A_3, \cdots \in \mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Example 1: Coin toss:

- $\Omega = \{H, T\}$.
- $\mathcal{F} = \{\phi, \{H\}, \{T\}, \ \Omega\}$.

Example 2: Die Roll

- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $\mathcal{F} = \{\phi, \{2, 4, 6\}, \{1, 3, 5\}, \Omega\}$.

Axiomatic definition of Probability

- Let R be a random experiment whose sample space is Ω. Let $\mathcal{F}$ is a $\sigma-$ field ($\sigma-$ algebra) defined on Ω.
- $\mathcal{F}$ can also be called as **EVENTS SPACE**.
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a Probability space. $\mathbb{P}$ is a set function from $\mathcal{F}$ to $[0, 1]$ satisfying the following three axioms:
  - $\mathbb{P}(\Omega) = 1$.
  - $\mathbb{P}(A) \geq 0$, $A \in \mathcal{F}$.
  - If $\{A_1, A_2, ...\} \subseteq \mathcal{F}$ is a countable collection of mutually disjoint events. Then

$$\mathbb{P}\left(\bigcup_{i=0}^{\infty} A_i\right) = \sum_{i=0}^{\infty} \mathbb{P}(A_i).$$

Some results based on Axiomatic definition

$(\Omega, \mathcal{F}, \mathbb{P})$ is a Probability space.

- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
- $\mathbb{P}(A \cup B \cup C) =$
  $\mathbb{P}(A)+\mathbb{P}(B)+\mathbb{P}(C)-\mathbb{P}(A\cap B)-\mathbb{P}(A\cap C)-\mathbb{P}(B\cap C)+\mathbb{P}(A\cap B\cap C)$.
- $\mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$..
- If $A \subseteq B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- Booles' inequality: $\mathbb{P}(A \cap B) \geq 1 - \mathbb{P}(A^c) - \mathbb{P}(B^c)$.

Problem 5

An engineer applies for a job in two firms, say **GOOGLE** and **NVIDIA**. The probability of his/her selection in **GOOGLE** is 0.8 and rejection in **NVIDIA** is 0.6. The probability of his/her rejection in at least one firm is 0.7. What is the probability that he/she will be selected in at least one of the firms.

Solution.

Let G : Selection in GOOGLE and N : Selection in NVIDIA. Given

$$\mathbb{P}(G) = 0.8, \qquad \mathbb{P}(N^c) = 0.6 \Rightarrow \mathbb{P}(N) = 1 - 0.6 = 0.4,$$

and

$$\mathbb{P}(G^c \cup N^c) = 0.7.$$

By De Morgan's law $(G^c \cup N^c)^c = G \cap N$, hence

$$\mathbb{P}(G \cap N) = 1 - \mathbb{P}(G^c \cup N^c) = 1 - 0.7 = 0.3.$$

The probability of being selected in *at least one* firm is

$$\mathbb{P}(G \cup N) = \mathbb{P}(G) + \mathbb{P}(N) - \mathbb{P}(G \cap N) = 0.8 + 0.4 - 0.3 = 0.9.$$

Newton–Pepys Problem

Samuel Pepys asked **Isaac Newton** which of the following three events is more likely when a fair die is rolled:

A: At least one "six" appears in 6 rolls of the die.

B: At least two "sixes" appear in 12 rolls of the die.

C: At least three "sixes" appear in 18 rolls of the die.

Which of the three events has the greatest probability?

Solution.

Total outcomes for n rolls: 6^n.

$$\mathbb{P}(A) = 1 - \mathbb{P}(\text{No six appear}) = 1 - \frac{5^6}{6^6} \approx 0.665102.$$

$$\mathbb{P}(B) = 1 - \Big[\mathbb{P}(\text{No six appear}) + \mathbb{P}(\text{One six appear})\Big] = 1 - \frac{5^{12}}{6^{12}} - \binom{12}{1}.\frac{5^{11}}{6^{12}} \approx 0.618667.$$

$$\mathbb{P}(C) = 1 - \Big[\mathbb{P}(\text{No six appear}) + \mathbb{P}(\text{One six appear}) + \mathbb{P}(\text{Two six appear})\Big]$$

$$= 1 - \frac{5^{18} + \binom{18}{1} \cdot 5^{17} + \binom{18}{2} \cdot 5^{16}}{6^{18}} \approx 0.597346.$$

Therefore, $\mathbb{P}(A) > \mathbb{P}(B) > \mathbb{P}(C)$, so event A is most likely.

THANK YOU.

Probability & Statistics (MA 2103)

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Geometric Probability

- Let R be a random experiment whose sample space Ω is a **Line segment**, **Area of a region**, **Volume**, ect.
- We are interested in the probability that a randomly chosen point in Ω falls in $A \subseteq \Omega$.
- If Ω and A have well defined finite measures (Length, Area, Volume), we define

$$\mathbb{P}(A) = \frac{\text{Measure of } A}{\text{Measure of } \Omega}.$$

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Problem 1: If a point is randomly chosen in from the interval [0, 10]. What is the probability that it is between 2 and 8.

Solution: The experiment is choosing a point randomly at random from $[0, 10]$. The sample space is

$$\Omega = [0, 10].$$

The length of Ω is 10. Let E denote the event "the chosen point lies between 2 and 8." Then

$$E = [2, 8].$$

The length of E is 8.

$$\mathbb{P}(E) = \frac{\text{length of } E}{\text{length of } \Omega} = \frac{6}{10} = 0.6.$$

Problem 2

A point is selected at random from within a circle of radius 2 units. What is the probability that the point lies closer to the center than to the circumference?

Solution: The experiment is choosing a point randomly from the disk of radius 2, say, centered at the origin. The sample space is

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq 4\}.$$

The area of Ω is: $\pi \cdot 2^2 = 4\pi$. Let E denote the event "the chosen point is closer to the center than to the circumference." If a point has distance b from the center, its distance to the circumference is $2 - b$. The condition "closer to the center than to the circumference" is

$$b < 2 - b \Longrightarrow b < 1.$$

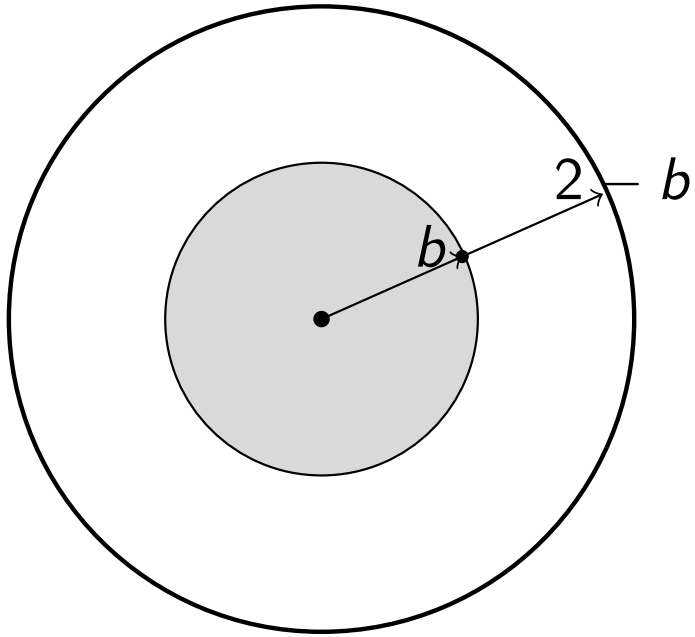
Hence E is the disk of radius 1 centred at the origin:

$$E = \{(x, y) : \ x^2 + y^2 \leq 1\}.$$

The area of E is π.

$$\mathbb{P}(E) = \frac{\text{area of } E}{\text{area of } \Omega} = \frac{\pi}{4\pi} = \frac{1}{4}.$$

Problem 2: Illustration



Problem 3

From the unit square, a point is selected randomly. Find the probability that this point lies within the unit circle.

Solution: The experiment is choosing a point uniformly at random from the unit square

$$\Omega = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

The area of the sample space is 1. The unit circle (radius 1, centred at the origin) is

$$x^2 + y^2 \leq 1.$$

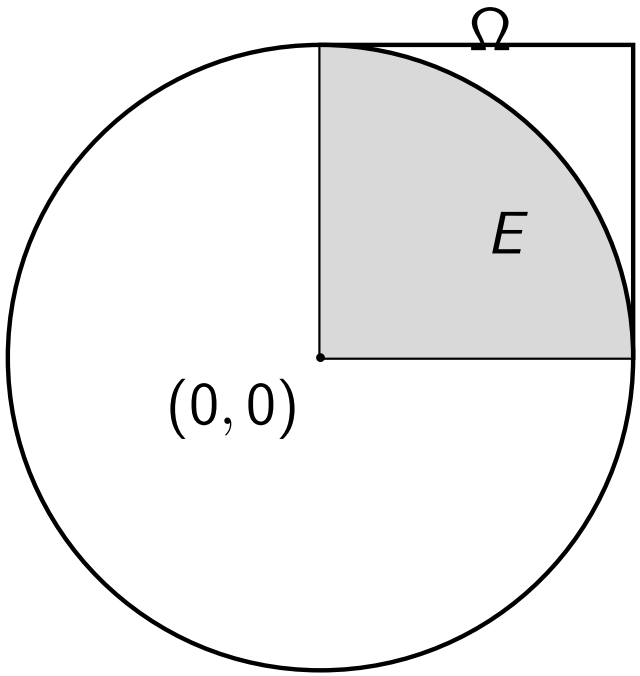
Within the unit square $[0, 1] \times [0, 1]$, the relevant event set is the quarter circle:

$$E = \{(x, y) : x^2 + y^2 \leq 1,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

The area of E is one quarter of the area of the unit circle: $\frac{\pi}{4}$. The probability is given by the ratio

$$\mathbb{P}(E) = \frac{\text{area of } E}{\text{area of } \Omega} = \frac{\frac{\pi}{4}}{1} = \frac{\pi}{4} \approx 0.785.$$

Problem 3: Illustration



Monte Carlo estimation of π, using $\mathbb{P}(E)$ in Problem 3

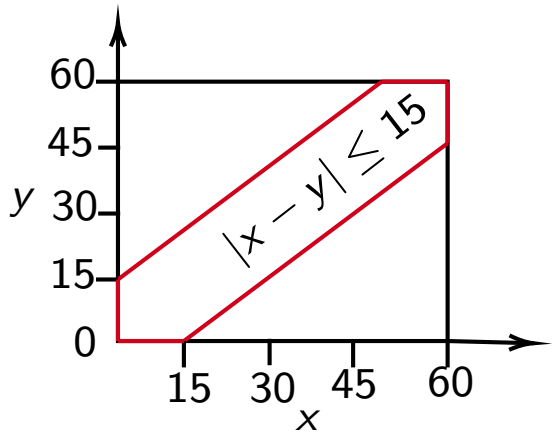
```
1  import random, matplotlib.pyplot as plt, numpy as np
2
3  N=1000; inside=0; est=[]; xin,yin,xout,yout,lin,lout=[],[],[],[],[],[]
4
5  for i in range(1,N+1):
6      x,y=random.random(),random.random()
7      if x**2+y**2<=1: inside+=1; xin+=[x]; yin+=[y]; lin+=[str(i)]
8      else: xout+=[x]; yout+=[y]; lout+=[str(i)]
9      est.append(4*inside/i)
10
11 fig,axs=plt.subplots(1,2,figsize=(12,6))
12 theta=np.linspace(0,np.pi/2,200)
13 axs[0].plot(np.cos(theta),np.sin(theta),'k--')
14 axs[0].scatter(xin,yin,c='r'); axs[0].scatter(xout,yout,c='b')
15 for (x,y,l) in zip(xin,yin,lin): axs[0].text(x+0.02,y,l,fontsize=6,c='r')
16 for (x,y,l) in zip(xout,yout,lout): axs[0].text(x+0.02,y,l,fontsize=6,c='b')
17 axs[0].set_xlim(0,1); axs[0].set_ylim(0,1); axs[0].set_aspect('equal')
18
19 axs[1].plot(range(1,N+1),est,marker='.')
20 axs[1].axhline(y=np.pi,c='r',ls='--',label='True    '); axs[1].legend()
21 plt.tight_layout(); plt.show()
```

Problem 4

Two friends plan to meet at the college cafeteria between 2:00 p.m. and 3:00 p.m., without fixing the exact minute. Each arrives uniformly at random during the hour and will wait at most 15 minutes for the other. Find the probability that they meet.

Solution: Let x be the arrival minute of friend A and y be the arrival minute of friend B. The sample space is the square: $\Omega = \{(x, y) : 0 \leq x \leq 60,\ 0 \leq y \leq 60\}$. The area of Ω is: $60 \times 60 = 3600$. They meet exactly when the difference in arrival times is at most 15 minutes, i.e.

$$E = \{(x, y) : |x - y| \leq 15,\ 0 \leq x, y \leq 60\}.$$



Geometrically E is the diagonal band inside the square between the lines $y = x + 15$ and $y = x - 15$. The area of E is 1575.

$$\mathbb{P}(E) = \frac{\text{area of } E}{\text{area of } \Omega} = \frac{1575}{3600} = \frac{7}{16}.$$

Problem 5

A rod of length "a" is broken randomly into three pieces. What is the probability that these pieces can form a triangle?

Solution: Let x and y denote the positions of the two break points along the rod. Since both break points can be anywhere along the rod independently:

$$0 \leq x \leq a, \quad 0 \leq y \leq a.$$

Hence, the sample space is the square $\Omega = \{(X, Y) : 0 \leq x \leq a,\ 0 \leq y \leq a\}$, with area a^2. The three pieces are

$$L_1 = \min(x, y), \quad L_2 = |x - y|, \quad L_3 = a - \max(x, y).$$

The pieces form a triangle if the triangle inequalities hold:

$$L_1 + L_2 > L_3, \quad L_2 + L_3 > L_1, \quad L_1 + L_3 > L_2.$$

If $\min(x, y) = x$, the above inequalities are equivalent to

$$x < a/2, \quad y - x < a/2, \quad y > a/2.$$

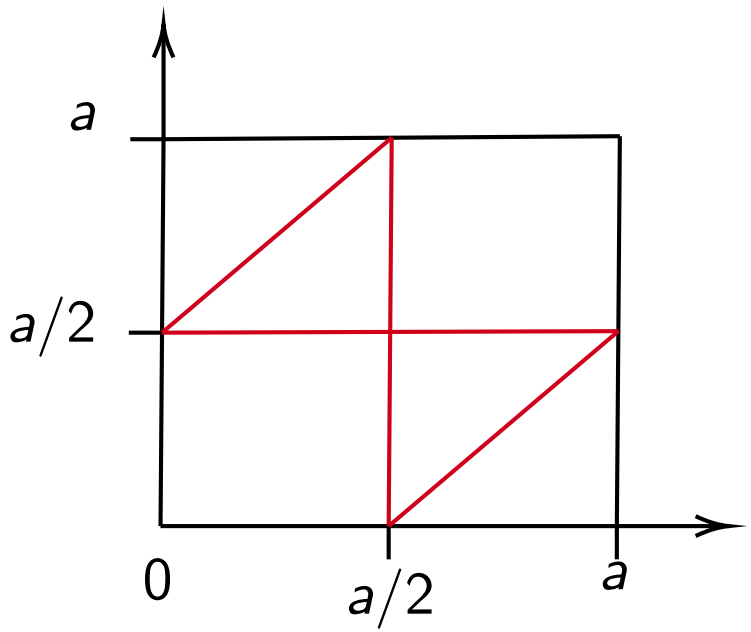
Similarly, if $\min(x, y) = y$,

$$x > a/2, \quad x - y < a/2, \quad y < a/2.$$

These inequalities simplify to the region

$$E = \{(X, Y) : x < a/2,\ y - x < a/2,\ \text{and}\ y > a/2 \quad || \quad x > a/2,\ x - y < a/2,\ \text{and}\ y < a/2\}.$$

Problem 5 Continue...



$$\text{area of } E = a^2 - 2 \cdot \frac{a^2}{8} = \frac{a^2}{4}.$$

$$\mathbb{P}(E) = \frac{\text{area of } E}{\text{area of } \Omega} = \frac{a^2/4}{a^2} = \frac{1}{4}.$$

Problem 6

A window is fitted with parallel iron rods, each spaced a distance of "$2a$" units apart. A student drops "a" pencil of length a randomly onto the window. The pencil always lands within the window frame (never outside). If the pencil touches a rod, the attempt is considered unsuccessful. What is the probability that the student is successful, i.e., the pencil does not touch any rod?

Solution: Let r = distance from pencil center to nearest rod, and θ = angle pencil makes with rods.

$$0 \leq r \leq a, \quad 0 \leq \theta \leq \pi$$

Sample space area: $a \cdot \pi$.

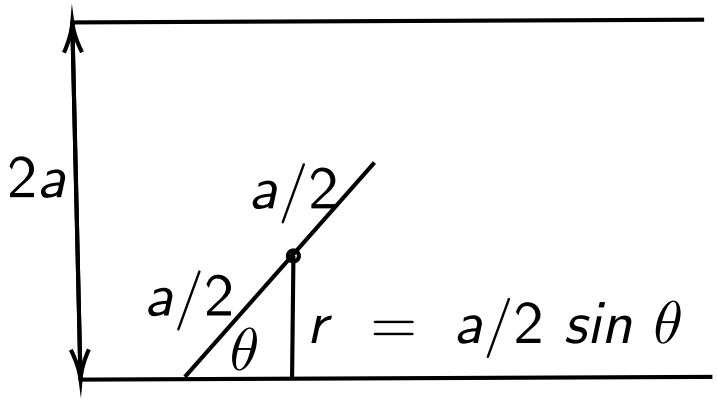
E: Pencil does not touch any rod. E happens if

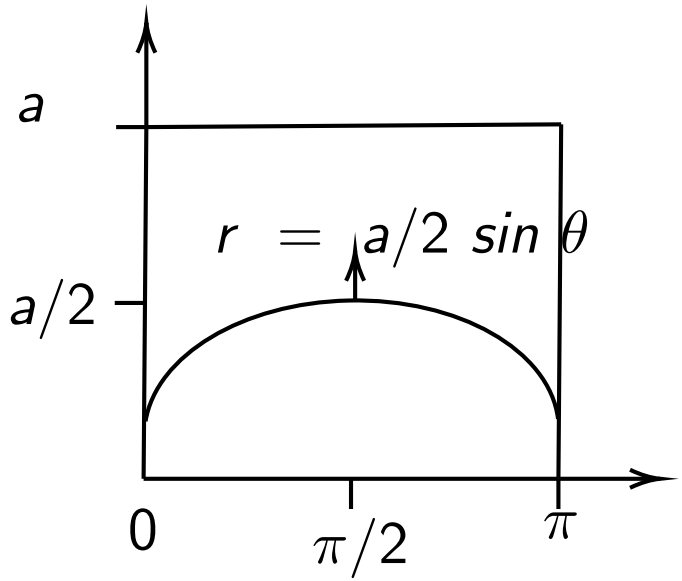
$$r \geq \frac{a}{2} \sin\theta.$$

Event region:

$$E = \{(r, \theta) : r \geq a/2 \sin\theta\}.$$

Problem 6 Continue...





$$\mathbb{P}(\text{success}) = \frac{\text{area of event}}{\text{area of sample space}} = \frac{a.\pi - \int_0^{\pi} \left(\frac{a}{2}\sin\theta\right) d\theta}{a \cdot \pi}$$

$$= \frac{(\pi - 1)}{\pi}.$$

THANK YOU.